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LAMINAR FREE CONVENTION BOUNDARY-LAYER IN THREE-DIMENSIONAL SYSTEMS

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Abstract—The series of several variables has been applied to the solutions of the boundary-layer equations of free convection in laminar three-dimensional systems. Numerical computation of the solutions has been investigated for the case of free convection over an inclined circular cylinder. The temperature profiles calculated from the first five terms of the series are compared with the experimental data. Some other possibilities of application of the results are mentioned.

NOMENCLATURE

a, exponent in expansion of the principal functions defined by equation (20);
f, g, non-dimensional stream functions defined in equation (9);
f_j, g_j, stream functions, coefficients in expansion of f and g in terms of ξ;
$f_{j,n}, g_{j,n}$, stream functions, coefficients in expansion of f_j and g_j in terms of ζ;
$\bar{g}$, acceleration vector;
$\bar{g}_x, \bar{g}_z$, local components of the acceleration vector in x and z-directions respectively;
Gr, Grashof number defined by

$$Gr = \frac{g\beta(T_0 - T_\infty)l^3}{\nu^2};$$

h, local heat transfer coefficient;
i,j,k, integer numbers;
$K, \bar{K}$, configuration functions, see equation (5);
$K_n, \bar{K}_n$, coefficients in expansion of K in terms of ζ;
l characteristic length;
m, n, integer numbers;
Nu, Nusselt number defined by equation (57);
Pr, Prandtl number defined by $Pr = \frac{\nu}{\alpha}$;
P, Q, principal functions defined in equation (17);
p, q, exponents in expansion of the configuration function S, see equation (18);
R, principal function defined in equation (17);
R_j, coefficients in expansion of R in terms of ξ;
r, radius of cylinder;
S, configuration function in x-direction;
T, temperature;
T_0, surface temperature;
T_∞, ambient temperature;
u, v, w, velocity components in x, y and z-directions respectively;
$\bar{u}, \bar{v}, \bar{w}$, non-dimensional velocity components defined by

$$\bar{u} = \frac{u}{(\nu/l)\,Gr^{1/2}}, \quad \bar{v} = \frac{v}{(\nu/l)\,Gr^{1/4}},$$

$$\bar{w} = \frac{w}{(\nu/l)\,Gr^{1/2}};$$

x, y, z, orthogonal coordinates;
$\bar{x}, \bar{y}, \bar{z}$, non-dimensional coordinates defined by

$$\bar{x} = \frac{x}{l}, \quad \bar{y} = Gr^{1/4}\frac{y}{l}, \quad \bar{z} = \frac{z}{l};$$

Greek symbols

α, thermal diffusivity;
β, thermal expansion coefficient;
γ, inclination angle of cylinder;
ζ, η, independent variables defined by equation (8);
θ, dimensionless local temperature defined by equation (9);
θ_j, coefficients in expansion of θ in terms of ξ;
$\theta_{j,n}$, coefficients in expansion of θ_j in terms of ζ;
λ, thermal conductivity;
ν, kinematic viscosity;
ξ, independent variable defined by equation (8);
$\bar{\phi}$, non-dimensional stream function defined in equation (7);
φ, circumferencial angle, x/r;
ψ, non-dimensional stream function defined in equation (7).

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1. INTRODUCTION

IN TWO-DIMENSIONAL geometries, for laminar free convection heat transfer from isothermal walls has been extensively treated with success by a number of investigators [1–4]. The application of Acrivos method [5], which is very useful in the prediction of the heat transfer coefficient of laminar two-dimensional free convection for large Prandtl numbers, to the three-dimensional systems has been proposed by Stewart [6, 7]. This work is the only investigation to the author's knowledge which had dealt theoretically with the problem of the free convection over three-dimensional bodies. As in the case of Acrivos solution for two-dimensional systems, however, at intermediate and small Prandtl numbers the Stewart solution can not be applied without a further analysis.

In the present study, an alternative approach to predict the coefficient of heat transfer, therefore the temperature and the velocity fields, for laminar free convection over three-dimensional bodies of arbitrary shape to a fluid of any Prandtl number, is presented. It is based on the simple notion that the solution of the partial differential equations of several variables function can be expressed in a series form of these variables. This viewpoint has been used by Duric [8, 9] to obtain the solution of the problem of unsteady incompressible laminar boundary-layers on two-dimensional bodies of arbitrary shape in the form of universal functions with respect to the body contour.

To examine the usefulness and limitations of the present study, the solution has been applied to the case of free convection on an inclined circular cylinder, in order to attempt comparison with the experimental data measured by Deluche [10].

2. DEVELOPMENT OF BASIC EQUATIONS

Consider the transfer of heat from a solid surface of uniform temperature T_0 to an infinite ambient fluid of undisturbed temperature T_∞, by steady laminar free convection. All fluid properties will be assumed to be constants except for the density changes which give the buoyancy terms in the momentum equations.

Let x and z denote the orthogonal coordinates in the wall surface with the origin at the stagnation point, and y denote the coordinate which is perpendicular to the wall. For free convection on two dimensional or axially symmetrical bodies only one component of the acceleration vector is taken into account in the boundary-layer equations, and it depends in general only on one space coordinate in the wall surface. The two components of velocity depend on two space coordinates. In the case of a three-dimensional free convection boundary-layer the flow of fluid is generated by the two components of the acceleration which can depend on two space coordinates in the wall surface each, and the flow within the boundary-layer possesses all three velocity components which, moreover, depend on all three space coordinates in the general case. If u, v and w denote these three velocity vector components in the direction of x, y and z respectively, the boundary-layer equations of continuity, motion and energy are, then, as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\bar{g}_x\beta(T-T_\infty)+v\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$v\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\bar{g}_z\beta(T-T_\infty)+v\frac{\partial^2 w}{\partial y^2} \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\alpha\frac{\partial^2 w}{\partial y^2}. \tag{4}$$

The x and z components of the acceleration vectors g_x and g_z, as have been mentioned earlier, depend in general on x and z, but in the present study we consider only a particular case, when $g_x(x)$ and $g_z(z)$. For convenience of further discussion, let us write

$$\bar{g}(x,z)=\bar{g}\,S(x)\bar{K}(z);\quad \bar{g}_z(z)=\bar{g}\,K(z). \tag{5}$$

The boundary conditions appropriate to the problem are

$$\begin{aligned} &y=0,\quad u=v=w=0,\quad T=T_0\\ &y\to\infty,\quad u,v,w\to 0,\quad T\to T_\infty. \end{aligned} \tag{6}$$

If the dimensionless stream functions which satisfy identically the continuity equation (1) are defined as

$$\bar{u}=\frac{\partial\bar{\psi}}{\partial\bar{y}};\quad \bar{w}=\frac{\partial\bar{\phi}}{\partial\bar{y}};\quad \bar{v}=-\left(\frac{\partial\bar{\psi}}{\partial\bar{x}}+\frac{\partial\bar{\phi}}{\partial\bar{z}}\right), \tag{7}$$

using the coordinate transformation

$$\xi=\int_0^x[S(t)]^{1/3}\mathrm{d}t;\quad \zeta=\bar{z};\quad \eta=\frac{[S(\xi)]^{1/3}\bar{y}}{(\frac{4}{3}\xi)^{1/4}(\frac{4}{3}\zeta)^{1/4}}, \tag{8}$$

and introducing new dependent variables f, g and θ such that

$$f(\xi,\eta,\zeta)=\frac{\bar{\psi}(\bar{x},\bar{y},\bar{z})}{(\frac{4}{3}\xi)^{3/4}(\frac{4}{3}\zeta)^{3/4}};\quad g(\xi,\eta,\zeta)=\frac{(\frac{4}{3}\xi)^{1/4}\bar{\phi}(\bar{x},\bar{y},\bar{z})}{[S(\xi)]^{1/3}(\frac{4}{3}\zeta)^{3/4}};$$

$$\theta(\xi,\eta,\zeta)=\bar{T}(\bar{x},\bar{y},\bar{z}), \tag{9}$$

one finds

$$\bar{u}=[S(\xi)]^{1/3}(\tfrac{4}{3}\zeta)^{1/2}(\tfrac{4}{3}\xi)^{1/2}f' \tag{10}$$

$$\bar{v}=-\frac{[S(\xi)]^{1/3}(\frac{4}{3}\zeta)^{3/4}}{(\frac{4}{3}\xi)^{1/4}}\left[f+(\tfrac{4}{3}\xi)\frac{\partial f}{\partial\xi}+\frac{\eta f'}{3}\left\{\frac{(\frac{4}{3}\xi)}{S(\xi)}\frac{\mathrm{d}[S(\xi)]}{\mathrm{d}\xi}-1\right\}+\frac{g}{(\frac{4}{3}\zeta)}+\frac{\partial g}{\partial\zeta}-\frac{\eta g'}{4\zeta}\right] \tag{11}$$

$$\bar{w}=\frac{[S(\xi)]^{2/3}(\frac{4}{3}\zeta)^{1/2}}{(\frac{4}{3}\xi)^{1/2}}g'. \tag{12}$$

The equations resulting from the transformation are:

$$f'''+gf''-\tfrac{2}{3}g'f'+(\tfrac{4}{3}\zeta)\{ff''-\tfrac{4}{3}P(\xi)f'^2\}+\bar{K}(\zeta)\theta=(\tfrac{4}{3}\zeta)\left\{\frac{\partial(f',g)}{\partial(\zeta,\eta)}+(\tfrac{4}{3}\xi)\frac{\partial(f',f)}{\partial(\xi,\eta)}\right\} \tag{13}$$

$$g'''+gg''-\tfrac{2}{3}g'^2+(\tfrac{4}{3}\zeta)\{fg''-\tfrac{4}{3}Q(\xi)f'g'\}+R(\xi)K(\zeta)\theta=(\tfrac{4}{3}\zeta)\left\{\frac{\partial(g',g)}{\partial(\zeta,\eta)}+(\tfrac{4}{3}\xi)\frac{\partial(g',f)}{\partial(\xi,\eta)}\right\} \tag{14}$$

$$Pr^{-1}\theta''+\{g+(\tfrac{4}{3}\zeta)f\}\theta'=(\tfrac{4}{3}\zeta)\left\{\frac{\partial(\theta,g)}{\partial(\zeta,\eta)}+(\tfrac{4}{3}\xi)\frac{\partial(\theta,f)}{\partial(\xi,\eta)}\right\}, \tag{15}$$

with

$$\begin{aligned}&\eta=0,\quad f=f'=g=g'=0,\quad \theta=1\\&\eta\to\infty,\quad f',g'\to 0,\quad \theta\to 0.\end{aligned} \tag{16}$$

In the forgoing expressions, the prime denotes derivative with respect to η, and $\partial(\,,)/\partial(x,\eta)$ is the Jacobian. The transformed configuration functions $P(\xi)$, $Q(\xi)$ and $R(\xi)$ are given by:

$$P(\xi)=\frac{1}{2}+\frac{1}{3}\frac{\xi}{S(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}[S(\xi)];\quad Q(\xi)=2\{P(\xi)-\tfrac{3}{4}\};\quad R(\xi)=\frac{4}{3}\frac{\xi}{[S(\xi)]^{4/3}}. \tag{17}$$

In order to employ Goertler type series [11], we assume that the configuration function $S(\bar{x})$ has the expansion form:

$$S(\bar{x})=\bar{x}^{p}\sum_{j=0}^{\infty}S_j\bar{x}^{qj}, \tag{18}$$

where p and q are positive real numbers. It can be demonstrated that the transformed configuration functions $P(\xi)$, $Q(\xi)$ and $R(\xi)$ which according to Goertler terminology can be also called the principal functions, take the following series forms:

$$P(\xi)=\sum_{j=0}^{\infty}P_j\xi^{aj};\quad Q(\xi)=\sum_{j=0}^{\infty}Q_j\xi^{aj};\quad R(\xi)=\sum_{j=0}^{\infty}R_j\xi^{aj}, \tag{19}$$

where P_0, Q_0, R_0 and a are obtained to be:

$$\begin{aligned}&P_0=\frac{3}{2}\left(\frac{p+1}{p+3}\right);\quad Q_0=3\left(\frac{p+1}{p+3}\right)-\frac{3}{2};\\&R_0=\begin{cases}0 & \text{for } p\neq 1\\ \dfrac{4}{S_0(p+3)} & \text{for } p=1\end{cases};\quad a=\frac{3q}{p+3}.\end{aligned} \tag{20}$$

One can show, upon putting $p=0$, $q=1$ and $p=1$, $q=2$ in the expression of P_0 and a, that they correspond identically to sharp and round-nosed cylinders, respectively, in the two-dimensional case as has been mentioned and discussed earlier by Saville and Churchill in [3]. Before we come to the further discussion, it should be noted that from the substitutions of $S(\bar{x})$ and ξ into $R(\xi)$ in equation (17), q in (18) should be restricted. The restriction is:

$$q=1-p,\quad \text{for}\quad p\neq 1. \tag{21}$$

According to the expansions of the principal functions, we assume for the solutions of (13), (14) and (15) satisfying (16), convergent series of the form

$$f(\xi,\eta,\zeta) = \sum_{j=0}^{\infty} f_j(\eta,\zeta)\xi^{aj}; \quad g(\xi,\eta,\zeta) = \sum_{j=0}^{\infty} g_j(\eta,\zeta)\xi^{aj};$$

$$\theta(\xi,\eta,\zeta) = \sum_{j=0}^{\infty} \theta_j(\eta,\zeta)\xi^{aj}. \tag{22}$$

Upon substituting (19) and (22) into (13), (14) and (15), collecting and equating to zero the coefficient of each power of ξ, the following groups of differential equations are obtained,

$$f_0''' + g_0 f_0'' - \tfrac{2}{3} g_0' f_0' + (\tfrac{4}{3}\zeta)\{f_0 f_0'' - \tfrac{4}{3}P_0 f_0''\} + \bar{K}(\zeta)\theta_0 = (\tfrac{4}{3}\zeta)\frac{\partial(f_0', g_0)}{\partial(\zeta,\eta)} \tag{23}$$

$$g_0''' + g_0 g_0'' - \tfrac{2}{3} g_0'^2 + (\tfrac{4}{3}\zeta)\{f_0 g_0'' - \tfrac{4}{3}Q_0 f_0' g_0'\} + R_0 K(\zeta)\theta_0 = (\tfrac{4}{3}\zeta)\frac{\partial(g_0', g_0)}{(\zeta,\eta)} \tag{24}$$

$$Pr^{-1}\theta_0'' + \{g_0 + (\tfrac{4}{3}\zeta)f_0\}\theta_0' = (\tfrac{4}{3}\zeta)\frac{\partial(\theta_0, g_0)}{\partial(\zeta,\eta)}, \tag{25}$$

for the higher orders are given by,

$$\begin{aligned} &f_j''' + g_0 f_j'' - \tfrac{2}{3}(g_0' f_j' + f_0' g_j') + f_0'' g_j + \bar{K}(\zeta)\theta_j \\ &\quad + (\tfrac{4}{3}\zeta)\Big\{ f_0 f_j'' + (1 + \tfrac{4}{3}aj) f_0'' f_j - \tfrac{4}{3}(2P_0 + aj) f_0' f_j' \\ &\quad - \Big(g_0' \frac{\partial f_j'}{\partial\zeta} + g_j' \frac{\partial f_0'}{\partial\zeta} - f_0'' \frac{\partial g_j}{\partial\zeta} - f_j'' \frac{\partial g_0}{\partial\zeta} \Big)\Big\} = L_{j-1} \end{aligned} \tag{26}$$

$$\begin{aligned} &g_j''' + g_0 g_j'' - \tfrac{4}{3} g_0' g_j' + g_0'' g_j + R_0 K(\zeta)\theta_j \\ &\quad + (\tfrac{4}{3}\zeta)\Big\{ f_0 g_j'' + (1 + \tfrac{4}{3}aj) g_0'' f_j - \tfrac{4}{3}(Q_0 + aj) f_0' g_j' - \tfrac{4}{3}Q_0 g_0' f_j' \\ &\quad - \Big(g_0' \frac{\partial g_j'}{\partial\zeta} + g_j' \frac{\partial g_0'}{\partial\zeta} - g_0'' \frac{\partial g_j}{\partial\zeta} - g_j'' \frac{\partial g_0}{\partial\zeta} \Big)\Big\} = M_{j-1} \end{aligned} \tag{27}$$

$$\begin{aligned} &Pr^{-1}\theta_j'' + g_0\theta_j' + \theta_0' g_j + (\tfrac{4}{3}\zeta) \\ &\quad \Big\{ f_0\theta_j' + (1 + \tfrac{4}{3}aj)\theta_0' f_j - \tfrac{4}{3}aj f_0' \theta_j \\ &\quad - \Big(\theta_0 \frac{\partial g_j'}{\partial\zeta} + \theta_j \frac{\partial g_0'}{\partial\zeta} - \theta_0' \frac{\partial g_j}{\partial\zeta} - \theta_j' \frac{\partial g_0}{\partial\zeta} \Big)\Big\} = N_{j-1}, \end{aligned} \tag{28}$$

$$\begin{aligned} L_{j-1} = &\sum_{i=0}^{j=1} (\tfrac{2}{3} g_i' f_{j-i}' - g_i f_{j-i}'') + (\tfrac{4}{3}\zeta)\Big[\tfrac{4}{3}\Big\{ P_j f_0'^2 + \sum_{i=1}^{j-1} (P_i f_0' f_{j-i}' + (P_0 + ai) f_i' f_{j-i}') \\ &+ \sum_{i=1}^{j-1}\sum_{k=1}^{j-i} P_i f_k' f_{j-k-i}' \Big\} - \sum_{i=1}^{j-1}\Big((1 + \tfrac{4}{3}ai) f_i f_{j-i}'' + g_i' \frac{\partial f_{j-i}'}{\partial\zeta} - f_{j-i}'' \frac{\partial g_i}{\partial\zeta} \Big)\Big] \end{aligned} \tag{29}$$

$$\begin{aligned} M_{j-1} = &\sum_{i=1}^{j-1} (\tfrac{2}{3} g_i' g_{j-i}' - g_i g_{j-i}'') + (\tfrac{4}{3}\zeta)\Big[\tfrac{4}{3}\Big\{ Q_j f_0' g_0' + \sum_{i=1}^{j-1} (Q_i f_0' g_{j-i}' \\ &+ (Q_0 + ai) g_i' f_{j-i}') + \sum_{i=1}^{j-1}\sum_{k=1}^{j-i} Q_i f_k' g_{j-k-i}' \Big\} - \sum_{i=1}^{j-1}\Big((1 + ai) f_i g_{j-i}'' \\ &+ g_i' \frac{\partial g_{j-}'}{\partial\zeta} - g_{j-i}'' \frac{\partial g_i}{\partial\zeta} \Big)\Big] - K(\zeta)\Big(R_j\theta_0 + \sum_{i=1}^{j-1} R_i\theta_{j-i} \Big) \end{aligned} \tag{30}$$

$$N_{j-1} = -\sum_{i=1}^{j-1} g_i'\theta_{j-i}' + (\tfrac{4}{3}\zeta)\Big[\sum_{i=1}^{j-1}\Big\{ \tfrac{4}{3}ai\theta_i f_{j-i}' - (1 + \tfrac{4}{3}ai) f_i \theta_{j-i}' + \theta_i \frac{\partial g_{j-i}'}{\partial\zeta} - \theta_{j-i}' \frac{\partial g_i}{\partial\zeta} \Big\}\Big]. \tag{31}$$

The boundary conditions are:

$$\begin{aligned} &\eta = 0, \quad f_j = f_j' = g_j = g_j' = 0, \quad \text{for} \quad j = 0,1,2,3,\ldots \\ &\theta_0 = 1, \quad \theta_j = 0, \quad \text{for} \quad j = 1,2,3,\ldots \\ &\eta \to \infty, \quad f_j', g_j', \theta_j \to 0, \quad \text{for} \quad j = 0,1,2,3,\ldots \end{aligned} \tag{32}$$

Now, in order to reduce equations (23)–(28) into the ordinary differential equations, the configuration function in the direction of z is assumed to have the expansion form:

$$K(\zeta)=\sum_{i=0}^{\infty} K_n\zeta^n\delta,=\bar{K}(\zeta)=\sum_{i=0}^{\infty}\bar{K}_n\zeta^n \tag{33}$$

Accordingly, we suggest the following expansions for $f_j(\eta,\zeta)$, $g_j(\eta,\zeta)$ and $\theta_j(\eta,\zeta)$ ($j = 0, 1, 2, 3, \ldots$):

$$f_j(\eta,\zeta)=\sum_{n=0}^{\infty} f_{j,n}(\eta)\zeta^n;\quad g_j(\eta,\zeta)=\sum_{n=0}^{\infty} g_{j,n}(\eta)\zeta^n;\quad \theta_j(\eta,\zeta)=\sum_{n=0}^{\infty}\theta_{j,n}(\eta)\zeta^n. \tag{34}$$

When equations (33) and (34) are substituted into (23)–(28), using the same procedure as in reducing (23)–(28) from (13)–(15), we obtain, for $j = n = 0$,

$$f'''_{0,0}+g_{0,0}f''_{0,0}-\tfrac{2}{3}g'_{0,0}f'_{0,0}+\bar{K}_0\theta_{0,0}=0 \tag{35}$$

$$g'''_{0,0}+g_{0,0}g''_{0,0}-\tfrac{2}{3}g'^2_{0,0}+R_0K_0\theta_{0,0}=0 \tag{36}$$

$$Pr^{-1}\theta''_{0,0}+g_{0,0}\theta'_{0,0}=0, \tag{37}$$

for $j = 0$, $n = 1, 2, 3, \ldots$

$$f'''_{0,n}+g_{0,0}f''_{0,n}-\tfrac{2}{3}(1+2n)g'_{0,0}f'_{0,n}-\tfrac{2}{3}f'_{0,0}g'_{0,n}+(1+\tfrac{4}{3}n)f''_{0,0}g_{0,n}+\bar{K}_0\theta_{0,n}=L_{0,n-1} \tag{38}$$

$$g'''_{0,n}+g_{0,0}g''_{0,n}-\tfrac{4}{3}(1+n)g'_{0,0}g'_{0,n}+(1+\tfrac{4}{3}n)g''_{0,0}g_{0,n}+R_0K_0\theta_{0,n}=M_{0,n-1} \tag{39}$$

$$Pr^{-1}\theta''_{0,n}+g_{0,0}\theta'_{0,n}-\tfrac{4}{3}ng'_{0,0}\theta_{0,n}+(1+\tfrac{4}{3}n)\theta'_{0,0}g_{0,n}=N_{0,n-1}, \tag{40}$$

for $j = 1, 2, 3, \ldots, n = 0$,

$$f'''_{j,0}+g_{0,0}f''_{j,0}+\tfrac{2}{3}(g'_{0,0}f'_{j,0}-f'_{0,0}g'_{j,0})+f''_{0,0}g_{j,0}+\bar{K}_0\theta_{j,0}=L_{j-1,0} \tag{41}$$

$$g'''_{j,0}+g_{0,0}g''_{j,0}-\tfrac{4}{3}g'_{0,0}g'_{j,0}+g''_{0,0}g_{j,0}+R_0K_0\theta_{j,0}=M_{j-1,0} \tag{42}$$

$$Pr^{-1}\theta''_{j,0}+g_{0,0}\theta'_{j,0}+\theta'_{0,0}g_{j,0}=N_{j-1,0}, \tag{43}$$

and for $j = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$,

$$f'''_{j,n}+g_{0,0}f''_{j,n}-\tfrac{2}{3}(1+2n)g'_{0,0}f'_{j,n}-\tfrac{2}{3}f'_{0,0}g'_{j,n}+(1+\tfrac{4}{3}n)f''_{0,0}g_{j,n}+\bar{K}_0\theta_{j,n}=L^{j-1,n}_{j,n-1} \tag{44}$$

$$g'''_{j,n}+g_{0,0}g''_{j,n}-\tfrac{4}{3}(1+n)g'_{0,0}g'_{j,n}+(1+\tfrac{4}{3}n)g''_{0,0}g_{j,n}+R_0K_0\theta_{j,n}=M^{j-1,n}_{j,n-1} \tag{45}$$

$$Pr^{-1}\theta''_{j,n}+g_{0,0}\theta'_{j,n}-\tfrac{4}{3}n\theta_{0,0}g'_{j,n}+(1+\tfrac{4}{3}n)\theta'_{0,0}g_{j,n}=N^{j-1,n}_{j,n-1}, \tag{46}$$

where

$$L_{0,n-1}=\tfrac{4}{3}\Big[\tfrac{4}{3}P_0f'_{0,0}f'_{0,n-1}-f_{0,0}f''_{0,n-1}+\sum_{m=1}^{n-1}\{\tfrac{4}{3}P_0f'_{0,m}f'_{0,n-m-1}=-K_m\theta_{0,n-m}$$
$$-(\tfrac{3}{4}+m)g_{0,m}f''_{0,n-m}+(\tfrac{1}{2}+m)f'_{0,m}g'_{0,n-m}-f_{0,m}f''_{0,n-m-1}\}\Big] \tag{47}$$

$$M_{0,n-1}=\tfrac{4}{3}\Big[\tfrac{4}{3}Q_0f'_{0,0}g'_{0,n-1}-\tfrac{3}{4}R_0K_n\theta_{0,0}-f_{0,0}g''_{0,n-1}+\sum_{m=1}^{n-1}\{\tfrac{4}{3}Q_0f'_{0,m}g'_{0,n-m-1}-\tfrac{3}{4}R_0K_m\theta_{0,n-m}$$
$$+(\tfrac{1}{2}+m)g'_{0,m}g'_{0,n-m}-(\tfrac{3}{4}+m)g_{0,m}g''_{0,n-m}-f_{0,m}g''_{0,n-m}\}\Big] \tag{48}$$

$$N_{0,n-1}=\tfrac{4}{3}f_{0,0}\theta'_{0,n-1}-\sum_{m=1}^{n-1}\{(1+\tfrac{4}{3}m)g_{0,m}\theta'_{0,n-m}+\tfrac{4}{3}(\theta_{0,m}g'_{0,n-m}-f_{0,m}\theta'_{0,n-m})\} \tag{49}$$

$$L_{j-1,0}=\sum_{i=1}^{j-1}(\tfrac{2}{3}g'_{i,0}f'_{j-i,0}-g_{i,0}f''_{j-i,0}) \tag{50}$$

$$M_{j-1,0}=\sum_{i=1}^{j-1}(\tfrac{2}{3}g'_{i,0}g'_{j-i,0}-g_{i,0}g''_{j-i,0}-K_0R_i\theta_{j-i,0})-K_0R_j\theta_{0,0} \tag{51}$$

$$N_{j-1,0}=-\sum_{i=1}^{j-1}g_{i,0}\theta'_{j-i,0} \tag{52}$$

$$
\begin{aligned}
L^{j-1,n}_{j,n-1} = {} & \tfrac{2}{3}\{f'_{j,0}g'_{0,n}+(1+2n)f'_{0,n}g'_{j,0}\}-(1+\tfrac{4}{3}n)g_{0,n}f''_{j,0}-f''_{0,n}g_{j,0} \\
& -\tfrac{4}{3}\{f_{0,0}f''_{j,n-1}+(1+\tfrac{4}{3}aj)f''_{0,0}f_{j,n-1}-\tfrac{4}{3}(2P_0+aj)f'_{0,0}f'_{j,n-1}\} \\
& +\sum_{m=1}^{n-1}\big[\tfrac{2}{3}(1+2m)(f'_{j,m}g'_{0,n-m}+f'_{0,m}g'_{j,n-m})-\tfrac{4}{3}\{f_{0,m}f''_{j,n-m-1} \\
& +(1+\tfrac{4}{3}aj)f''_{0,m}f_{j,n-m-1}-\tfrac{4}{3}(2P_0+aj)f'_{0,m}f'_{n-m-1}\}-\bar{K}_m\theta_{j,n-m} \\
& -(1+\tfrac{4}{3}m)(g_{0,m}f''_{j,n-m}+g_{j,m}f_{0,n-m})\big] \\
& +\sum_{m=0}^{n}\sum_{i=1}^{j-1}\big[(\tfrac{2}{3}g'_{i,m}f'_{j-i,n-m}-g_{i,m}f''_{j-i,n-m})+\tfrac{4}{3}(P_i f'_{0,m}f'_{j-i,n-m} \\
& -(1+\tfrac{4}{3}ai)f_{i,m}f''_{j-i,n-m}\big]+\tfrac{4}{3}\Big|\,\tfrac{4}{3}P_j\sum_{m=0}^{n-1}f'_{0,m}f'_{0,n-m-1} \\
& +\sum_{m=1}^{n}\sum_{i=1}^{j-1}m(f'_{j-i,m}g_{i,n-m}-g_{i,m}f''_{j-i,n-m})+\sum_{m=0}^{n-1}\sum_{i=1}^{j-1}\sum_{k=1}^{j-i}P_i f'_{k,m}f'_{j-k,n-m}
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
M^{j-1,n}_{j,n-1} = {} & \tfrac{4}{3}(1+n)g'_{0,n}g'_{j,0}-(1+\tfrac{4}{3}n)g_{0,n}g''_{j,0}-g''_{0,n}g_{j,0}-R_0K_n\theta_{j,0} \\
& -\tfrac{4}{3}\{f_{0,0}g'_{j,n-1}+(1+\tfrac{4}{3}aj)g''_{0,0}f_{j,n-1}-\tfrac{4}{3}(Q_0+aj)f'_{0,0}g'_{j,n-1} \\
& -\tfrac{4}{3}Q_0g'_{0,0}f'_{j,n-1}\}-\sum_{m=1}^{n-1}\big[(1+\tfrac{4}{3}m)(g_{0,m}g''_{j,n-m}+g_{j,m}g''_{0,n-m}) \\
& +\tfrac{4}{3}(1+m)g'_{0,m}g'_{j,n-m}+\tfrac{4}{3}mg'_{j,m}g'_{0,n-m}-R_0K_m\theta_{j,n-m} \\
& -\tfrac{4}{3}\{f_{0,m}g''_{j,n-m-1}+(1+\tfrac{4}{3}aj)g''_{0,m}f_{j,n-m-1}-\tfrac{4}{3}(Q_0+aj)f'_{0,m}g'_{j,n-m-1} \\
& -\tfrac{4}{3}Q_0g'_{0,m}f'_{j,n-m-1}\}\big]+\tfrac{16}{9}Q_j\sum_{m=0}^{n-1}f'_{0,m}g'_{0,n-m-1} \\
& +\sum_{m=0}^{n}\sum_{i=1}^{j-1}(\tfrac{2}{3}g'_{i,m}g'_{j-i,n-m}-g_{i,m}g''_{j-i,n-m}) \\
& +\sum_{m=0}^{n-1}\sum_{i=1}^{j-1}\big[\tfrac{16}{9}\{Q_if'_{0,m}g'_{j-i,n-m-1}+(Q_0+ai)g'_{i,m}f'_{j-i,n-m-1}\} \\
& -\tfrac{4}{3}(1+\tfrac{4}{3}ai)f_{i,m}g''_{j-i,n-m-1}\big]+\tfrac{4}{3}\sum_{m=1}^{n}\sum_{i=1}^{j-1}m(g'_{j-i}g'_{i,n-m} \\
& -g_{i,m}g''_{j-i,n-m})+\tfrac{4}{3}\sum_{m=0}^{n-1}\sum_{i=1}^{j-1}\sum_{k=1}^{j-i}Q_if'_{k,m}g'_{j-k-i,n-m-1}
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
N^{j-1,n}_{j,n-1} = {} & \tfrac{4}{3}ng'_{0,n}\theta_{j,0}-(1+\tfrac{4}{3}n)g_{0,n}\theta'_{j,0}-g_{j,0}\theta'_{0,n}-\tfrac{4}{3}\{f_{0,0}\theta'_{j,n-1} \\
& +(1+\tfrac{4}{3}aj)\theta'_{0,0}f_{j,n-1}-\tfrac{4}{3}ajf'_{0,n}\theta_{j,n-1}\} \\
& +\sum_{m=1}^{n-1}\big[\tfrac{4}{3}\{m(g'_{j,m}\theta_{0,n-m}+g'_{0,m}\theta_{j,n-m})-f_{0,m}\theta'_{j,n-m-1} \\
& -(1+\tfrac{4}{3}aj)f_{j,n-m-1}\theta'_{0,m}+\tfrac{4}{3}ajf'_{0,m}\theta_{j,n-m-1}\} \\
& -(1+\tfrac{4}{3}m)(g_{0,m}\theta'_{j,n-m}+g_{j,m}\theta'_{0,n-m})\big] \\
& +\sum_{m=0}^{n-1}\sum_{i=1}^{j-1}\big[\tfrac{4}{3}\{\tfrac{4}{3}aif'_{j-i,n-m-1}\theta_{i,m} \\
& -(1+\tfrac{4}{3}ai)f_{i,m}\theta'_{j-i,n-m-1}\}\big]-\sum_{m=0}^{n}\sum_{i=1}^{j-1}g_{i,m}\theta'_{j-i,n-m} \\
& +\tfrac{4}{3}\sum_{m=1}^{n}\sum_{i=1}^{j-1}m(g'_{j-i}\theta_{i,n-m}-g_{i,m}\theta'_{j-i,n-m}).
\end{aligned} \tag{55}
$$

The boundary conditions (32) become

$$
\begin{gathered}
\eta=0,\quad f_{j,n}=f'_{j,n}=g_{j,n}=g'_{j,n}=0 \\
\theta_{j,n}=\begin{cases}1 & \text{for}\quad j=n=0\\ 0 & \text{for}\quad j\neq n\neq 0\end{cases} \\
\eta\to\infty,\quad f'_{j,n},\quad g'_{j,n},\quad \theta'_{j,n}\to 0.
\end{gathered} \tag{56}
$$

Knowing the coefficients in the expansions of functions $P(\xi)$, $Q(\xi)$, $R(\xi)$ and $K(\zeta)$, the classical numerical integration methods of first order ordinary differential equations systems now can be applied. Of course for practical purposes, the computation of the first few terms of the solutions would be enough, as our experience shows in the case of two-dimensional systems.

The local heat transfer coefficient in terms of the Nusselt number can be obtained from the following expression:

$$Nu = \frac{hl}{\lambda(T_0 - T_\infty)} = -\frac{Gr^{1/4}[S(\xi)]^{1/3}}{(\frac{4}{3}\xi)^{1/4}(\frac{4}{3}\zeta)^{1/4}} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \theta'_{i,j}(0)\zeta^n \xi^{aj}. \tag{57}$$

3. APPLICATIONS

Without neglecting the possibilities of application of the theory to other surface configurations, the results will be applied here only to the case of free convection from the outer surface of an inclined circular cylinder placed in the gravitational acceleration field, since experimental data for this case are available for comparison. Under consideration is a circular cylinder having radius r, tilted with the inclination angle γ. When the coordinates x and z are taken respectively as circumferential and longitudinal coordinates, and upon selecting r as the characteristic length, we identify

$$P(\xi) = \frac{3}{4} - \left(\frac{3}{40}\right)\left(\frac{64}{27}\right)^{1/2} \xi^{3/2} + \ldots \tag{58}$$

$$Q(\xi) = -\left(\frac{3}{20}\right)\left(\frac{64}{27}\right)^{1/2} \xi^{3/2} + \ldots \tag{59}$$

$$R(\xi) = \frac{1}{\cos\gamma} + \frac{1}{5}\left(\frac{64}{27}\right)^{1/2} \frac{1}{[\cos\gamma]^{3/2}} \xi^{3/2} + \ldots \tag{60}$$

$$K(\zeta) = \sin\gamma, \quad \bar{K}(\zeta) = 1 \tag{61}$$

wherein, ξ is given by:

$$\xi = [\cos\gamma]^{1/3} \int_0^{\varphi} [\sin t]^{1/3} \, dt. \tag{62}$$

The first five terms of the series have been numerically calculated on a digital computer following the integration procedure of Runge–Kutta–Merson [12]. The curves illustrated in Figs. 1–3 are typical results obtained from numerical integrations. The computed temperature fields at a distance $z = r$ for $\varphi = 0$, 45, 90 and 135° under inclination angles $\gamma = 25$ and 45° with Prandtl number $Pr = 0.72$ are shown in Figs. 4–7 together with the experimental data of Deluche [10] for air. As it is seen, the agreement between the theoretical results and the experimental data is satisfactory. It should be noted that, for the calculations here, all the physical properties are evaluated at the film temperature, this conducing to the different value of Grashof number estimated in [10]. Moreover, since the measurements of temperature at distances less than 1.5 mm seem to be in error, they were not plotted.

Although it is expected that the series in terms of ζ is convergent only for $\zeta \leqslant 1$, the temperature fields have also been calculated for distances $z = 3r, 5r$ and the results agreed reasonably with the experimental evidence.

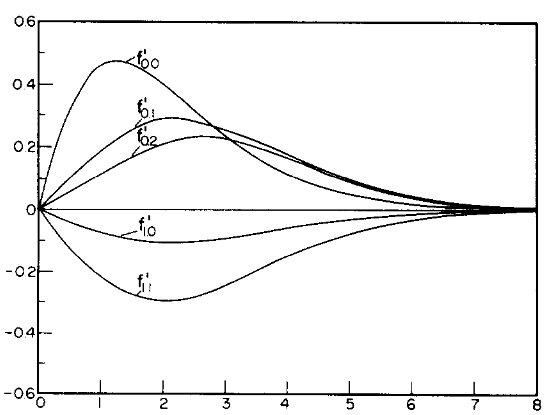

FIG. 1. The functional coefficients of the stream function $f(\xi, \eta, \zeta)$, for $\gamma = 45°$ and $Pr = 0.72$.

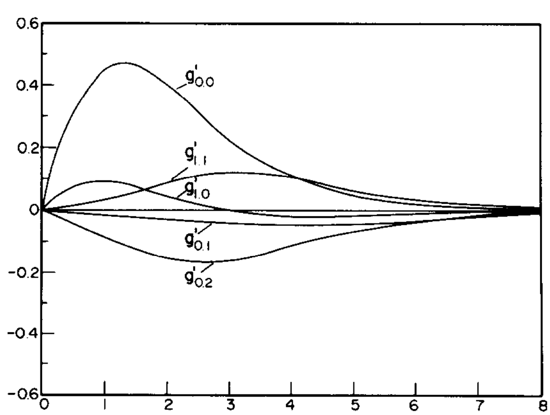

FIG. 2. The functional coefficients of the stream function $g(\xi, \eta, \zeta)$, for $\gamma = 45°$ and $Pr = 0.72$.

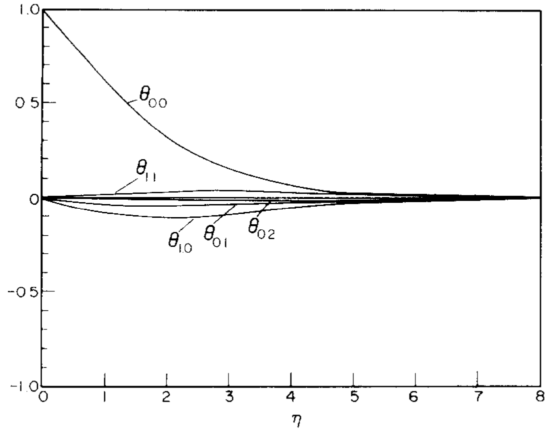

FIG. 3. The functional coefficients of the dimensionless temperature function $\theta(\xi, \eta, \zeta)$, for $\gamma = 45°$ and $Pr = 0.72$.

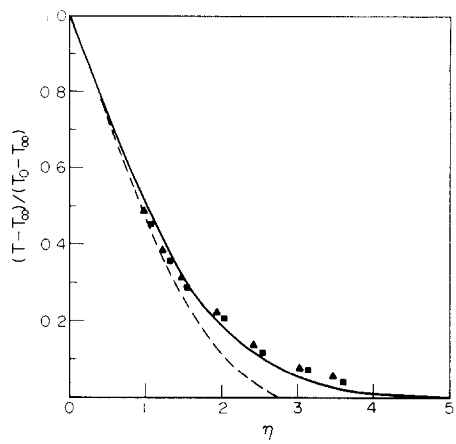

FIG. 6. Comparison of computed temperature profiles at $z = r$, $\varphi = 90°$, for $Pr = 0.72$, $Gr = 1.3 \times 10^6$ (----- $\gamma = 25°$, —— $\gamma = 45°$) with the experimental data (■ $\gamma = 25°$, ▲ $\gamma = 45°$).

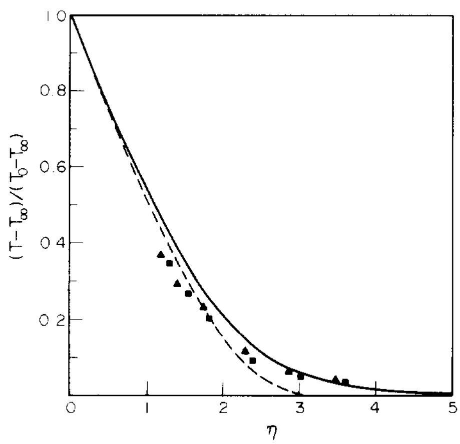

FIG. 4. Comparison of computed temperature profiles at $z = r$, $\varphi = 0°$, for $Pr = 0.72$, $Gr = 1.3 \times 10^6$ (----- $\gamma = 25°$, —— $\gamma = 45°$) with the experimental data (■ $\gamma = 25°$, ▲ $\gamma = 45°$).

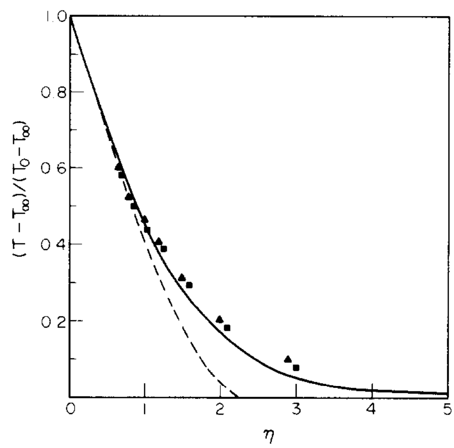

FIG. 7. Comparison of computed temperature profiles at $z = r$, $\varphi = 135°$, for $Pr = 0.72$, $Gr = 1.3 \times 10^6$ (----- $\gamma = 25°$, —— $\gamma = 45°$) with the experimental data (■ $\gamma = 25°$, ▲ $\gamma = 45°$).

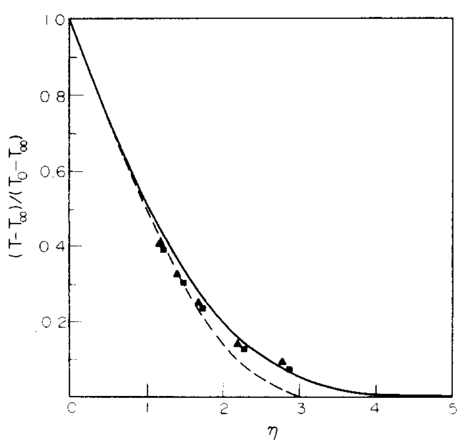

FIG. 5. Comparison of computed temperature profiles at $z = r$, $\varphi = 45°$, for $Pr = 0.72$, $Gr = 1.3 \times 10^6$ (---- $\gamma = 25°$, —— $\gamma = 45°$) with the experimental data (■ $\gamma = 25°$, ▲ $\gamma = 45°$).

4. DISCUSSION

The general solutions presented in the first part of this paper have a great potential of applications. It has been noted by the author that the series obtained can be applied to more complicated surface configurations such as inclined elliptical cylinders, vertical or inclined toruses etc. Furthermore, the method used here seems also to be applicable to the problems in hydrodynamic and forced convection boundary-layers of three-dimensional systems.

Although it has been shown that the application of the series for inclined circular cylinders verifies satisfactorily the temperature fields obtained from the experiment, it is however, desirable to have further verification to compare other results such as velocity profiles or other applications with the experimental results. Unfortunately, the author was unable to find them.

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COUCHE LIMITE DE CONVECTION NATURELLE LAMINAIRE DANS DES SYSTEMES TRIDIMENSIONNELS

Résumé—On propose l'utilisation de la série à plusieurs variables pour les solutions des équations de couche limite en convection naturelle dans des systèmes tridimensionnels. Les solutions obtenues ont été calculées numériquement dans le cas de la convection naturelle autour d'un cylindre incliné. Les profils de température calculés en prenant les cinq premiers termes sont comparés avec des résultats expérimentaux. Quelques autres possibilités d'application de la théorie sont citées.

GRENZSCHICHT BEI LAMINARER FREIER KONVEKTION IN DREIDIMENSIONALEN SYSTEMEN

Zusammenfassung—Reihen mit mehreren Variablen wurden zur Lösung der Grenzschichtgleichungen für die freie Konvektion in laminaren dreidimensionalen Systemen angewandt. Die numerische Berechnung der Lösungen für den Fall freier Konvektion über einem geneigten Kreiszylinder wurde untersucht. Die mit den ersten fünf Gliedern der Reihe berechneten Temperaturprofile werden mit experimentellen Daten verglichen. Einige andere Anwendungsmöglichkeiten der Ergebnisse werden erwähnt.

ЛАМИНАРНЫЙ СВОБОДНОКОНВЕКТИВНЫЙ СЛОЙ В ТРЁХМЕРНЫХ СИСТЕМАХ

Аннотация — Для решения уравнений пограничного слоя при наличии свободной конвекции в ламинарных трёхмерных системах используются ряды по нескольким переменным. Даётся оценка решений для случая свободной конвекции на наклонном круглом цилиндре. Температурные профили, рассчитанные по первым пяти членам ряда, сравниваются с экспериментальными данными. Указано на несколько возможных случаев использования результатов данной работы.